THE MATHEMATICS OF WINNING GAMES

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Introduction

Games have been a staple of modern entertainment for most of human history, and have grown quickly from straightforward games of logic into a countless quantity of radically different games. This has been influenced by a variety of factors including the level of enjoyment that can be found in games, and this level of enjoyment is at its peak when players defeat each other, specifically for the victor. Few can deny the satisfaction of success throughout any part of life, and this includes games. As an exercise in problem solving, computer science, and arithmetic skills, we examine four games and determine mathematically sound methods to play them, which give players the best possible chance of winning[1]. The four games which we examine in this paper are:

- Craps
- Blackjack
- In-Between
- Monopoly

In these games, we implement a variety of tools to grapple with how to play them. We use probability theory to analyze these games and create a program written in Python 3.7[2] to apply to the monopoly section specifically in order to get us accurate, analyzable results. The other purpose of this project was as a point of demonstration: it proves that math can be effectively applied to (more or less) useful scenarios.

[1] As chance is a major aspect in these games, it is important to note that no strategy is guaranteed to win. In this paper we only seek to outline ways to play which increase the likelihood of success.
[2] We modified the code to run more efficiently after generating our data and updated it to Python 3.8. This is the version of the code which is included here, although there is no functional difference between the two algorithms.

CHAPTER I

Craps

1.1 Introduction

Craps is a dice game in which players seek to make money by betting on the outcome of two dice. On some rolls, they make money and on other rolls, they lose their bet. The objective of this chapter is to define a mathematical approach to deal with this and determine a strategy that gives players using it a statistical advantage over their adversaries. Due to the high level of chance involved in craps, this is no small feat, and additionally, as this is a chance based game, no strategy is guaranteed to win with one hundred percent efficiency. Our goal is simply to give players their best possible chance at winning through mathematical and computational means. This approach implements the inherently different probabilities in the outcome of rolling two dice to determine how players should bet[3] and what probabilities they should bet on based on both the payout ratios of numbers rolled as well as their expected frequency.

1.2 Strategy

Given the immense amount of randomness in craps, the uninclined may believe that there is no optimal method of play, and that they should simply randomly assign their bets to numbers based on either the payout, or personal beliefs. Given the properties of a craps table, it is quite possible to construct a mathematical understanding of what is happening; in both the Come Out rolls[4] and in later betting. These are discussed in detail throughout this chapter, however a mathematical understanding of the game

[3] Due to the multitude of types of bets in craps, this paper will only examine the Pass Line bet, Don't pass bet, and the Place Win bet.
[4] It is assumed that players already have some understanding of craps terminology in this paper.

requires an understanding of the probabilities of rolling each number on a dice. These are shown in section 1.2.1[5].

1.2.1 Derivation of probabilities of rolling two dice

As each dice has six sides, the total number of combinations of numbers rolling from these two dice is finite. As there are six sides, and two dice the total number of combinations is $6 \times 6 = 36$. Therefore, we define a function $p(n)$ which calculates the probability of rolling a number n on two six sided dice as a fraction with a numerator equal to the possible number of combinations which sum to the number in question and a denominator of 36, that is: $p(n) = \frac{\textit{number of combinations summing to } n}{36}$. First, we calculate the number of combinations summing to each possible value of n for two dice[6].

$2 \in \{ (1, 1) \}$

$3 \in \{ (1, 2), (2, 1) \}$

$4 \in \{ (1, 3), (2, 2), (3, 1) \}$

$5 \in \{ (1, 4), (2, 3), (3, 2), (4, 1) \}$

$6 \in \{ (1, 5), (2, 4), (3, 3), (4, 2), (5, 1) \}$

$7 \in \{ (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1) \}$

$8 \in \{ (2, 6), (3, 5), (4, 4), (5, 3), (6, 2) \}$

$9 \in \{ (3, 6), (4, 5), (5, 4), (6, 3) \}$

$10 \in \{ (4, 6), (5, 5), (6, 4) \}$

$11 \in \{ (5, 6), (6, 5) \}$

$12 \in \{ (6, 6) \}$

It is worth noting that the notation used here is abnormal. For each value of n we express each possible sum as a set of ordered pairs, where the first term in each pair corresponds to the first die, and the

[5] I acknowledge that I am by no means the first person to prove the probabilities of each dice, and that my derivation is by no means unique.

[6] Given that the minimum number that can be rolled on each die is 1, the smallest value of n is 2. For the same reason, the greatest values of n is 12, as the maximum number rolled on each die is 6, and 6 + 6 = 12.

second term in the pair relates to the second die. The normal distribution of each n being rolled is presented in Figure 1. From this, we can apply our formula $p(n) = \frac{\text{number of combinations summing to } n}{36}$ to determine the probability of rolling each value of n. See Figure 2 for a graphic representation of the percentages.

$$p(2) = \tfrac{1}{36} = 2.\overline{777}\%$$

$$p(3) = \tfrac{2}{36} = 5.\overline{555}\%$$

$$p(4) = \tfrac{3}{36} = 8.\overline{333}\%$$

$$p(5) = \tfrac{4}{36} = 11.\overline{111}\%$$

$$p(6) = \tfrac{5}{36} = 13.\overline{888}\%$$

$$p(7) = \tfrac{6}{36} = 16.\overline{666}\%$$

$$p(8) = \tfrac{5}{36} = 13.\overline{888}\%$$

$$p(9) = \tfrac{4}{36} = 11.\overline{111}\%$$

$$p(10) = \tfrac{3}{36} = 8.\overline{333}\%$$

$$p(11) = \tfrac{2}{36} = 5.\overline{555}\%$$

$$p(12) = \tfrac{1}{36} = 2.\overline{777}\%$$

Figure 1: Number of Combinations summing to Values of n

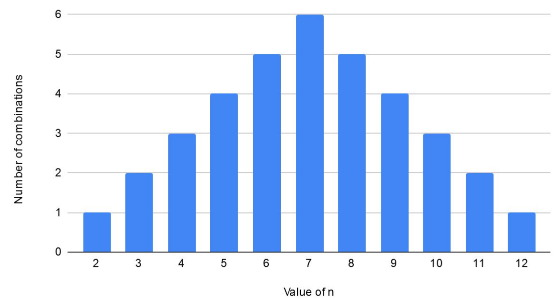
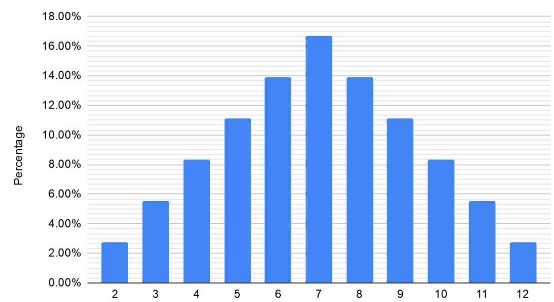


Figure 2: Percentage of Rolling each Value of n



1.2.2 General Strategy in Craps

We examine craps strategy first by examining the Come Out roll. There are six forms of bets that will be examined: the Pass Line bet, Don't Pass bet, Come and Don't Come bets, and the Place Win bet. These are by no means the only forms of bets in craps, however, they are regarded as some of the basic and common forms of bets. The strategy in this game is quite abstract; it cannot be affected by the players as it is just the rolls of the dice. For this reason, therefore, there is no strategy with a 100% success rate.

However, given the probabilities outlined in 1.2.1, we can order this game's chaos and give players their best chance at winning.

We begin by examining the Pass Line, and the Don't Pass bets. These bets are placed by the shooter (the player rolling) prior to making their Come Out roll. A Pass Line bet is a bet which pays out upon rolling a 7 or an 11. Meanwhile if a 2, 3, or 12 is rolled, the player forfeits their bet and the round is over. If they roll anything other than 2, 3, 7, 11, or 12, that value is called the point, and the game progresses to the next stage of betting. The near inverse of this is a Don't Pass bet, which pays out on a 2, 3, and loses on a 7, or an 11. Now, if a 12 is rolled the bet is mute and is pushed so there is no winner. Still, any other value rolled becomes the point value, but now upon rolling a 7, the player wins their bet, and loses if they roll the point. The first step in our mathematical analysis of our game features determining whether a player should bet on the Pass Line, or Don't Pass. This yields some simple computations:

$$\text{Pass Line odds} = 16.\overline{666} + 5.\overline{555} = 22.\overline{111}\%$$

$$\text{Don't Pass odds} = 2.\overline{777} + 5.\overline{555} = 8.\overline{222}\%$$

Despite that $22.\overline{111} > 8.\overline{222}$, it is more mathematically advantageous to make Don't Pass bets. This reason is discussed with Don't Come bets.

Next we shall consider the Come and Don't Come bets, these bets are functionally quite similar to the Pass Line and Don't Pass bet, with one main exception: they are placed after the establishment of the Come Out roll. On a Come bet, players win if the first roll after they place the Come bet is a 7 or an 11, and they lose the bet if a 2, 3, or 12 is rolled. If anything else is rolled, this becomes the players point, and they will win their wager if they roll this value. On the other hand, we have the Don't Come bet. If the player rolls a 2, 3, or 12 immediately after placing the Don't Come bet, they win their wager. If they roll a 7, or an 11 they lose their bet, and if they roll anything else, they establish the point. Once the point has been established, the bet will shift. Upon rolling a 7, the player will win their bet, and lose upon rolling the point. On the surface, players may seek to make Come bets for the same reason as we suggest making Pass Line bets over Don't Pass bets, i.e. that they almost triple the odds of it being rolled. However, we see:

$$\text{Odds of not setting point} = 2\left(2.\overline{777}\%\right) + 2\left(5.\overline{555}\%\right) + 16.\overline{666}\% = 33.\overline{333}\%$$

Therefore, there is a $100\% - 33.\overline{333}\% = 66.\overline{666}\%$ chance of setting a point. Given that the Don't Come bet will pay out on a 7 after setting a point, the odds of a Don't Come bet paying out at some point is effectively much greater than the odds of winning a Come bet, which after the the point is set is only a maximum of $13.\overline{888}\%$, and in practice is generally less than this, versus the fixed $16.\overline{666}\%$ chance of winning the bet if the player makes a Don't Come bet. Additionally, the odds of losing a Don't Come bet is comparatively less than the odds of losing a Come bet, to be exact, it is maximum of $30.\overline{222}\%$ versus $33.\overline{333}\%$. Therefore, it is statistically much more advantageous for players to make Don't Come bets as they both are fixed to and will only pay out on one number, and this one number has a higher probability of being rolled then the maximum probability of being rolled for a Come bet.

Next we consider the Place Win bet. This bet is very simple to understand and very easy to apply computationally. Essentially, the Place Win bet functions by betting that a certain number will be rolled before a 7 is rolled. To make these bets more appealing, there are risk to reward ratios attached to each of the numbers which it is possible to bet on: these ratios are: 6:7 for 6 and 8, 5:7 for 5 and 9, and 5:9 for 4 and 10. What these risk reward ratios mean is that for the ratio X:Y, for every X amount of money which a player risks, they can make Y amount of money. Taking these ratios into account, we can determine a how much a player is expected to make per roll, versus how much they should lose per roll based on the following formula:

$$P = A\,R$$

Where P is the expected profit per roll, R is the return per roll, and A is the chance of rolling. This formula is covered in more detail in Chapter III. Additionally, we can use this formula to determine the rate of losing money, in total giving us the expecting winnings per role, which is calculated as:

$$P = A\,R - 0.\overline{333}\,(Zx) + Y$$

Where Z is the expected loss of money, x is the amount of the minimum bet being put in, and Y is the amount of money that the player has. For the sake of argument, we set $Y = 10$. Solving this for each ratio, we derive:

For 6 and 8: $P = 0.13\overline{8}\,(7\,x) - 0.\overline{333}\,(6\,x) + 10$

For 5 and 9: $P = 0.\overline{111}\,(7\,x) - 0.\overline{333}\,(5\,x) + 10$

For 4 and 10: $P = 0.8\overline{33}\,(9\,x) - 0.\overline{333}\,(5\,x) + 10$

All of these equations simplify to the linear equations:

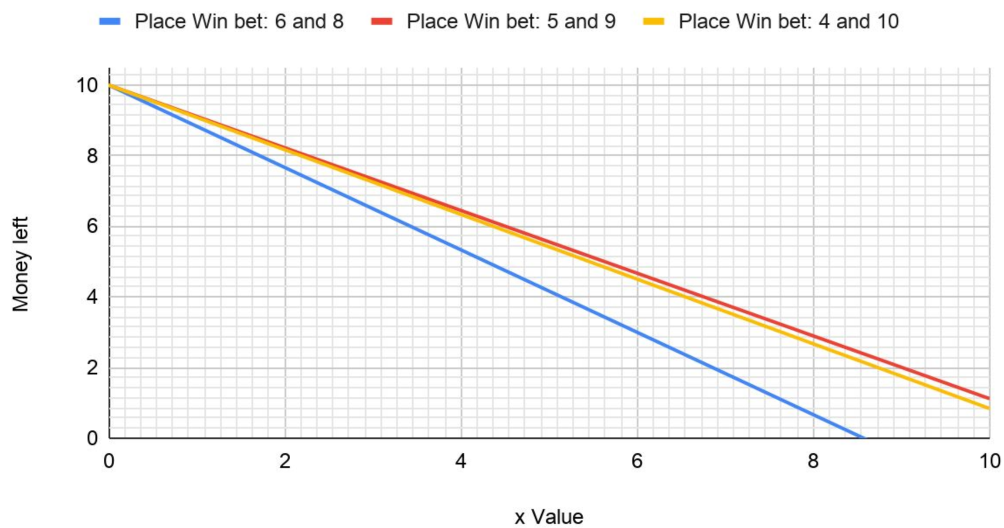
For 6 and 8: $P = -1.1\overline{66}\,x + 10$

For 5 and 9: $P = -0.\overline{888}\,x + 10$

For 4 and 10: $P = -0.91\overline{6}\,x + 10$

Given that all of these equations have a negative slope, the rate of change in profit is actually negative, meaning that these bets are quite inefficient,= and lose money over time. We show these in Figure 3-A.

Figure 3-A: Loss of Money by Place Win bet



We can use the same method used for the Place Win bets to directly compare the efficiency of the Come and Don't Come bets, however because the payout ratio for these bets is 1:1, it simplifies to[7]:

$$P = A\,R - 0.\overline{333}x$$

For Come bets:

Best case scenario: $P = 0.13\overline{8}\,x - 0.\overline{333}\,x + 10 = P = -0.19\overline{4}\,x + 10$

Worst case scenario: $P = 0.08\overline{3}\,x - 0.\overline{333}\,x + 10 = P = -0.25\,x + 10$

For Don't Come bets:

Best case scenario: $P = 0.1\overline{66}\,x - 0.25\,x + 10 = P = -0.08\overline{3}\,x + 10$

Worst case scenario: $P = 0.1\overline{66}\,x - 0.30\overline{2}\,x + 10 = P = -0.13\overline{5}\,x + 10$

Comparing these equations in Figure 3-B shows that the Don't Come bet is indeed the most efficient option for saving money, and that it loses players the least amount of money.

[7] As previously shown, for the Don't Come bet, there is a different probability of losing the bet, so that is used instead when applicable.

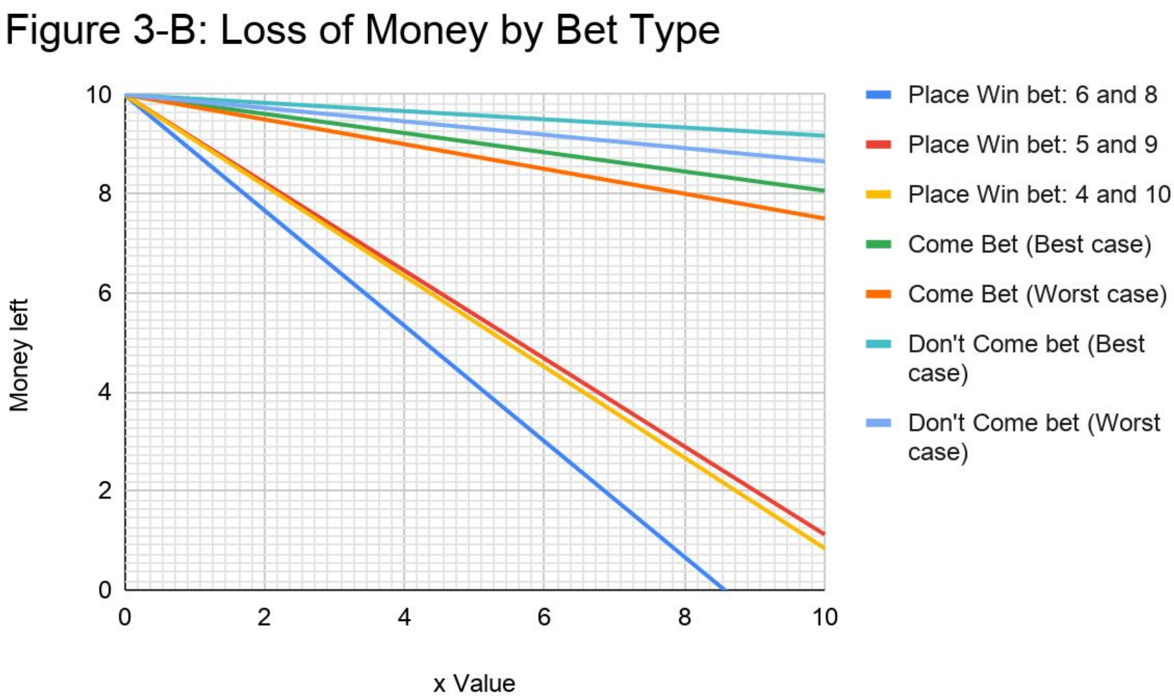


Figure 3-B also reaffirms a fact about gambling games: The house always wins, as all of the bets are expected to negatively affect the player. However, via applying mathematics to the problem, the impact of this is effectively reduced. We recommend a betting style focused around the Don't Come bet, as it gives players the likeliest way of making money, despite that the odds are still against them. In this way, it is feasible, although still less likely for players to profit playing craps.

CHAPTER II

Complete Information Card Games

2.1 Introduction

Card games can be categorized into two main fields: games with complete information and games with incomplete information. Complete information games are games in which players can see everything that other players have, for example Blackjack. On the other hand, games with incomplete information are games in which players lack certain key pieces of information, for example Poker, as each player has their own private hand which only they can see. In this chapter we derive a general formula for calculating the probability of drawing beneficial cards in certain games with complete information; specifically, we will be applying this formula to Blackjack and to In-Between, which we determine a more simplistic equation for. These games have very similar features meaning that there is almost no difference that needs to be made to the formula.

2.2 Derivation of Formulae

Let us consider the following: a player needs to draw an ace from a standard deck of playing cards, if they pick a random card from the deck, what is the chance that it is an ace? To solve this, we recall that the probability of an event happening, $q = \frac{Number\ of\ events}{Number\ of\ things}(100)$. Additionally, knowing that there are 52 cards in a deck, and 4 of these cards are aces, we calculate:

$$q = \tfrac{4}{52}(100)$$

$$= \qquad q = \tfrac{1}{13}(100)$$

$$= \qquad q \approx 7.7\%$$

What if we were only seeking to draw an ace of spades? Given that there is only 1 ace of spades in a deck, the probability is:

$$q = \tfrac{1}{52}(100)$$

$$= \qquad q \approx 1.9\%$$

These examples lead us to the general formula for calculating the odds of any card in a deck of cards, or any group of cards:

$$q = \frac{n}{52 - W}$$

Where n is the number of 'successes' or cards that do not immediately create a negative result, and W is the number of cards removed from a deck.

Indeed, these calculations are functional for any card. And by using them, we can create a strong approximation method for calculating the probability of drawing any card from a complete deck. Given that:

$$q = \frac{1}{52}(100) \approx 1.9\% \approx 2\%$$

We can calculate the probability of drawing any group of cards as 2 times the number of cards in this group, i.e. that

$$q \approx 0.02n\,(100)$$

$$\therefore q \approx 2n$$

2.3 Applied to Blackjack and In-Between

In Blackjack, players try to beat the dealer and get as close to 21 as possible without going over[8] (called going bust). By using this formula, players can easily calculate the odds of them going bust. as they can see each other player's cards, they can determine that the number of successes with the formula $q = \frac{n}{W}$. It is important to note that when playing Blackjack with more than one deck, this formula remains functional because the ratio to the number of a certain type of card and the number of cards in a deck remain the same.

[8] There are other rules which will not be covered here. We assume that readers already understand the playing of the game.

For In-Between, the second probability formula simplifies to $q = 2n$, instead of $q \approx 2n$. This is because in In-Between, there are always only two cards in play at the time of betting, thus[9]:

$$q = \frac{1}{52-2}n\,(100)$$

$$= \qquad q = \frac{1}{50}n\,(100)$$

$$= \qquad q = 2n$$

With this in mind, we can apply some simple analysis to In-Between[10], shown in Figure 4. It is important to note that while in Blackjack, kings, queens, and jacks are all valued at 10, in In-Between they are valued differently with jacks valued at 11, queens at 12 and kings at 14.

Figure 4: Success probabilities in In-Between

Number of cards between	Probability of success	Probability of failure
1	$q = 2\,(1 \times 4) = 8\%$	$100\% - 8\% = 92\%$
2	$q = 2\,(2 \times 4) = 16\%$	$100\% - 16\% = 84\%$
3	$q = 2\,(3 \times 4) = 24\%$	$100\% - 24\% = 76\%$
4	$q = 2\,(4 \times 4) = 32\%$	$100\% - 32\% = 68\%$
5	$q = 2\,(5 \times 4) = 40\%$	$100\% - 40\% = 60\%$
6	$q = 2\,(6 \times 4) = 48\%$	$100\% - 48\% = 52\%$
7	$q = 2\,(7 \times 4) = 56\%$	$100\% - 56\% = 44\%$
8	$q = 2\,(8 \times 4) = 64\%$	$100\% - 64\% = 36\%$
9	$q = 2\,(9 \times 4) = 72\%$	$100\% - 72\% = 28\%$
10	$q = 2\,(10 \times 4) = 80\%$	$100\% - 80\% = 20\%$
11	$q = 2\,(11 \times 4) = 88\%$	$100\% - 88\% = 12\%$

[9] In In-Between, given that there is always 4 of each type of card in a deck, the probability of drawing a successful card is always a multiple of 8, because the formula effectively becomes $q = 2 \times 4 \times n' = q = 8n'$, where n' is the number of types of cards in between the other cards.

[10] Unfortunately, we cannot perform a similar analysis on the game of Blackjack as the probability of drawing a card varies with the number of people playing. To accurately analyze this, we would need to judge all possible cards for all possible numbers of players.

Therefore, if a player is given two cards with and there are at least 7 cards with values in between them, or a difference between the card values of at least 8, the player is likely to have their next card fall in between these cards and thus should take this bet.

Chapter III

Monopoly

3.1 Introduction

Monopoly is a game which simulates a capitalist society, and in this society players compete to purchase property. Each property has a rent value assigned to it, this acts as a penalty for players who land on a property owned by another player; i.e. they must pay the property's owner an amount of money equal to the rent value. Players win Monopoly by bankrupting opponents through controlling properties and profiting when opponents land on these properties. To a naive player, an optimal strategy might appear to be saving their money until they land on the properties near the end of the loop, as these properties warrant higher rents. However this strategy is innately flawed as the properties near the end of the board are seldom landed on. Smarter players may seek to purchase every property that they can. This would ensure that they are controlling the most property. However, by not focusing on what they are buying, players lose the potential rewards of controlling a colour set, and tend to struggle later in the game as they lack the amount of money to pay off the higher rents of more developed colour sets with houses and hotels. Indeed, we observe that these are some of the most common strategies amongst Monopoly players.

In this paper, we find an optimal strategy for maximizing a player's income via analyzing the probabilities of landing on each square per turn. Our strategy has the distinct advantage over many other strategies, like the two discussed above, in that it focuses on the ratio of expected return from each property or colour set by the end of the players turn to their total investment into the property or colour set. This way, players can expect a maximal return per turn for a proportionally smaller investment. Another benefactor to this approach is that the results which we obtained do not match with the properties most commonly sought after by players in the early game. Meaning that it is more likely that opponents will not buy the properties which a player may require for this strategy to work. Thus, our strategy has an additional advantage over others.

3.2 Methodology and Implications

3.2.1 General

At first glance at a Monopoly board, it would appear that in a game each property is landed on about the same amount of times. This is not the case, due to certain squares that send you to other squares (Chance and Community Chest) or the rule that if any player rolls three consecutive doubles, they go to jail immediately. To account for this, we use a brute-force computer program which effectively simulates one million rolls on a board. The program then outputs each property as a fraction of one million. From this, we can derive the percentage which each square is landed on. The full code for this program is in Appendix B. The data presented is derived as the average of five trials of this code. See Figure 1 for the data sheet with each trial as well as the average from all trials.

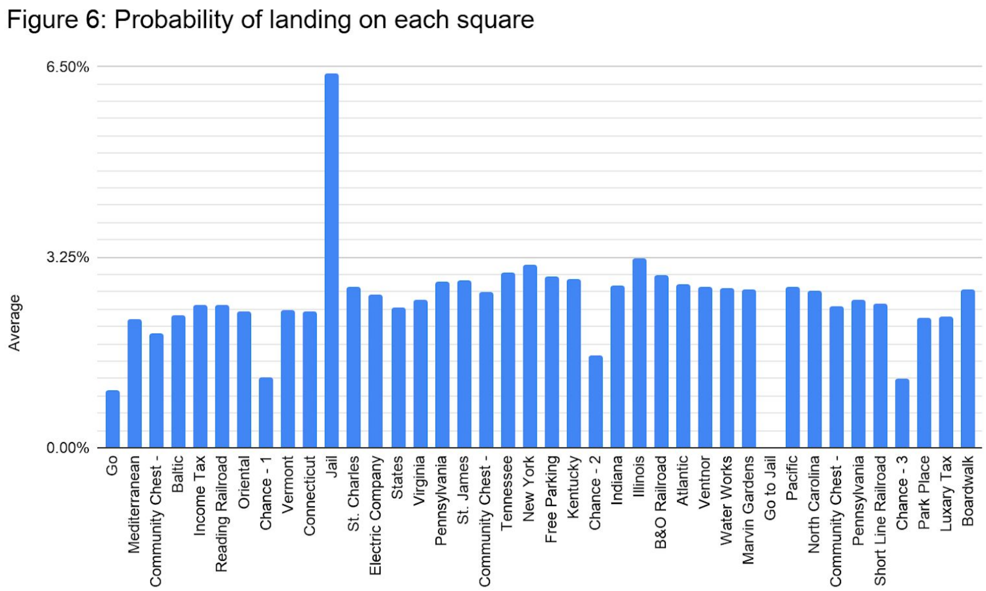
Figure 5: Results from code expressed as a percentage

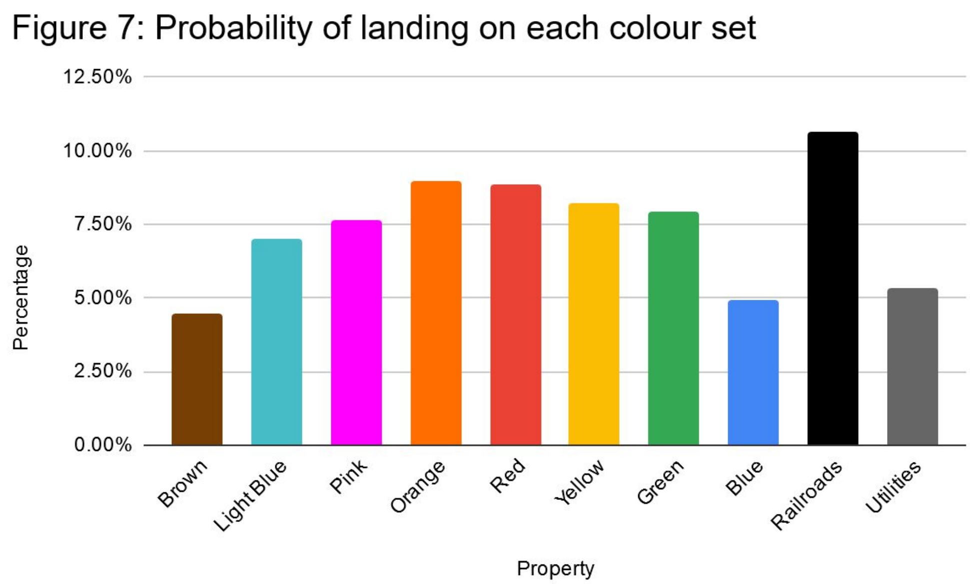
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Average
Go	0.99%	0.97%	0.96%	0.99%	0.97%	0.98%
Mediterranean	2%	2.20%	2.20%	2.18%	2.20%	2.19%
Community Chest - 1	1.98%	1.95%	1.96%	1.96%	1.97%	1.96%
Baltic	2.23%	2.26%	2.27%	2.27%	2.29%	2.26%
Income Tax	2.44%	2.41%	2.43%	2.42%	2.43%	2.43%
Reading Railroad	2.44%	2.47%	2.42%	2.45%	2.43%	2.44%
Oriental	2.33%	2.33%	2.31%	2.33%	2.31%	2.32%
Chance - 1	1.18%	1.19%	1.19%	1.19%	1.21%	1.19%
Vermont	2.33%	2.36%	2.37%	2.32%	2.36%	2.35%
Connecticut	2.32%	2.32%	2.32%	2.33%	2.30%	2.32%
Jail	6.44%	6.37%	6.41%	6.36%	6.34%	6.38%
St. Charles	2.73%	2.73%	2.76%	2.78%	2.76%	2.75%

Electric Company	2.60%	2.64%	2.62%	2.61%	2.64%	2.62%
States	2.43%	2.37%	2.37%	2.40%	2.38%	2.39%
Virginia	2.54%	2.53%	2.50%	2.52%	2.52%	2.52%
Pennsylvania Railroad	2.84%	2.80%	2.82%	2.82%	2.86%	2.83%
St. James	2.83%	2.86%	2.87%	2.86%	2.84%	2.85%
Community Chest - 2	2.64%	2.67%	2.66%	2.63%	2.64%	2.65%
Tennessee	2.99%	2.97%	3.00%	2.99%	2.98%	2.99%
New York	3.12%	3.12%	3.13%	3.12%	3.15%	3.13%
Free Parking	2.94%	2.88%	2.90%	2.92%	2.90%	2.91%
Kentucky	2.86%	2.91%	2.87%	2.88%	2.88%	2.88%
Chance - 2	1.59%	1.58%	1.58%	1.57%	1.58%	1.58%
Indiana	2.76%	2.73%	2.77%	2.77%	2.75%	2.76%
Illinois	3.22%	3.25%	3.23%	3.25%	3.25%	3.24%
B&O Railroad	2.95%	2.98%	2.94%	2.95%	2.93%	2.95%
Atlantic	2.81%	2.77%	2.78%	2.78%	2.80%	2.79%
Ventnor	2.75%	2.77%	2.75%	2.75%	2.75%	2.75%
Water Works	2.70%	2.71%	2.75%	2.72%	2.73%	2.72%
Marvin Gardens	2.69%	2.71%	2.70%	2.71%	2.69%	2.70%
Go to Jail	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Pacific	2.76%	2.74%	2.76%	2.75%	2.75%	2.75%
North Carolina	2.68%	2.67%	2.67%	2.70%	2.68%	2.68%
Community Chest - 3	2.40%	2.40%	2.44%	2.41%	2.42%	2.41%
Pennsylvania	2.53%	2.54%	2.53%	2.53%	2.55%	2.53%
Short Line Railroad	2.44%	2.46%	2.45%	2.46%	2.46%	2.45%
Chance - 3	1.19%	1.18%	1.18%	1.17%	1.19%	1.18%
Park Place	2.21%	2.20%	2.23%	2.22%	2.24%	2.22%

Luxury Tax	2.23%	2.22%	2.23%	2.25%	2.23%	2.23%
Boardwalk	2.69%	2.75%	2.71%	2.69%	2.69%	2.71%

We can express the data as a graph, giving us the probability of ending one's turn on each specific square[1] (Figure 6). By adding the probabilities of landing on each property set, we graph a comparison of each property by the frequency of which each colour set is landed on (Figure 7).





However, these graphs are useless for players who wish to determine the optimal strategy to play the game, as they only show part of the picture, and thus are not effective on their own. To develop more useful data with regards to this, we need to examine the ratio between a player's investment (defined here as the total amount of money spent on a property or colour set) and the investment's rent. By combining the rent, probability of landing on each square and the player's total investment, we get:

$$P = A\ R$$

Where P is the profit that a player wants to make (in this case the investment), A is the efficiency of the property, or amount of times that a player needs to land on the property, and R is the expected return per roll. Thus:

$$R = B\ X$$

$$\therefore\ P = A\ B\ X$$

Where B is the rent of the property, and X is the probability of landing on that property per turn. Values in this formula are given for P, B, X thus, we can algebraically derive values for A following:

$$P = A\ B\ X$$

$$= \frac{P}{B\,X} = A$$

However, there still is a problem. This formula will not hold for calculating the efficiency of a colour set, as there are multiple different values for *B* and *X*. To accomplish this, we average the values of *B*. For *n* number of turns, this adjustment yields[11]:

$$P = A\,X\left(\frac{B_1 + B_2 + \ldots + B_n}{n}\right)$$

$$= A = \frac{P}{X\left(\frac{B_1 + B_2 + \ldots + B_n}{n}\right)}$$

Applying these formulae, we generate results for finding the most efficient properties to own given the following scenarios:

1. A direct comparison of each property's efficiency (Figure 8).
2. No houses in play, comparing colour sets (Figure 9).
3. Comparing colour sets with one house per property (Figure 10).
4. Comparing colour sets with two houses per property (Figure 11).
5. Comparing colour sets with three houses per property (Figure 12).
6. Comparing colour sets with four houses per property (Figure 13).
7. Comparing colour sets with one hotel per property (Figure 14).

Via analyzing these graphs, we can determine that there are indeed colour sets with more value than others. It is quickly apparent that the Orange, Yellow, and Blue properties are the most advantageous to control early in the game, as they yield large sums of money for relatively small investments. However, as the game progresses and there are more houses in play, the cost of buying houses on the Yellow and Blue colour sets becomes out of hand, we begin to see this when there are three houses per property in the set, i.e. Figure 12. From then on, Orange and Red become the two 'best' colour sets to own, however, when there are hotels on each colour set, Light Blue surpasses Red as the second most efficient colour set. We observe that the difference between efficiency on owning 4 houses versus one hotel is not great.

[11] See Appendix B for a complete derivation of all values of *A*.

Therefore, in addition to buying the previously recognised colour sets, we recommend only putting 4 houses on each property, as it nearly maximizes one's own profit potential while detracting the most from opponents. I.e. this way the player monopolizes the houses making landing on other player's properties significantly less of a threat.

Figure 8: Hundreds of Turns until Profit (No Houses) By Property

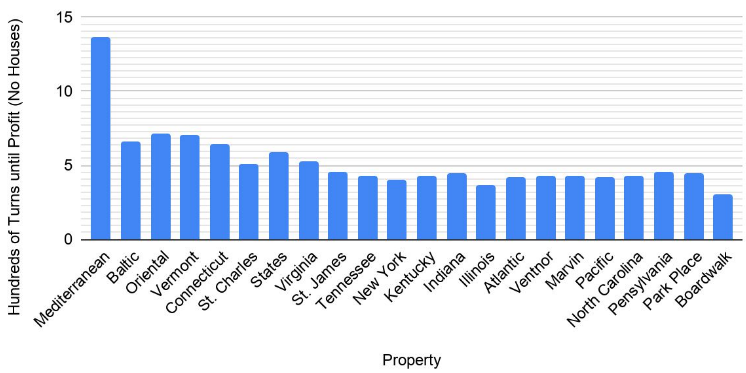


Figure 9: Hundreds of Turns Until Profit (No Houses) By Colour Set

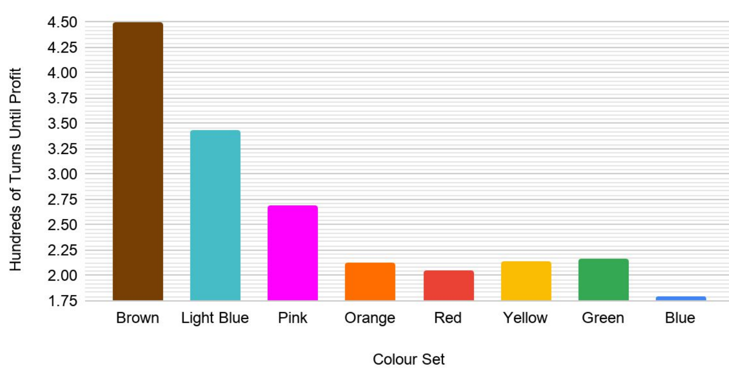


Figure 10: Hundreds of Turns Until Profit (One House) By Colour Set

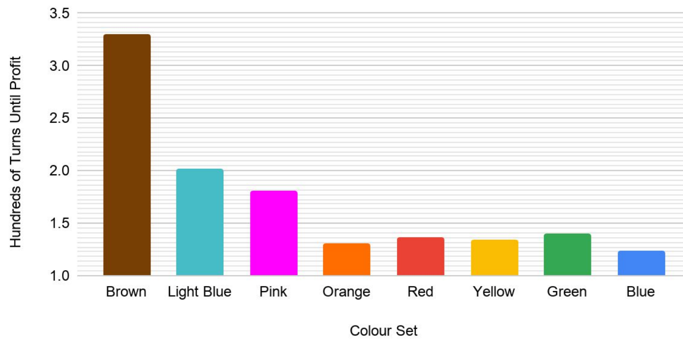


Figure 11: Tens of Turns until Profit (Two Houses) By Colour Set

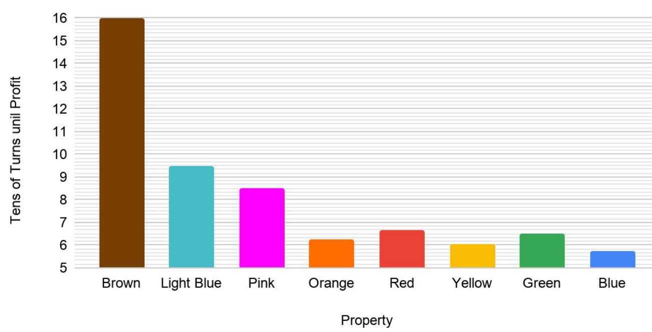


Figure 12: Tens of Turns until Profit (Three Houses) By Colour Set

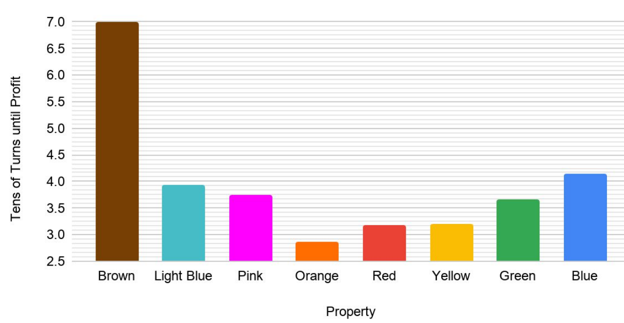


Figure 13: Tens of Turns until Profit (Four Houses) By Colour Set

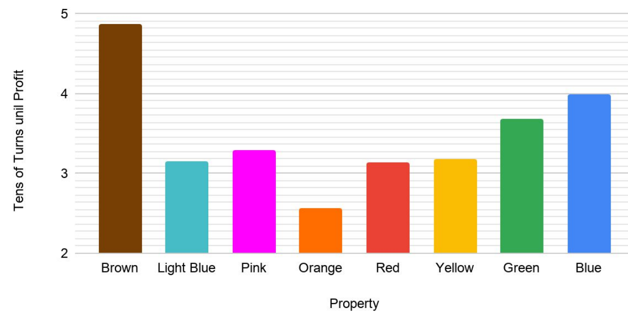
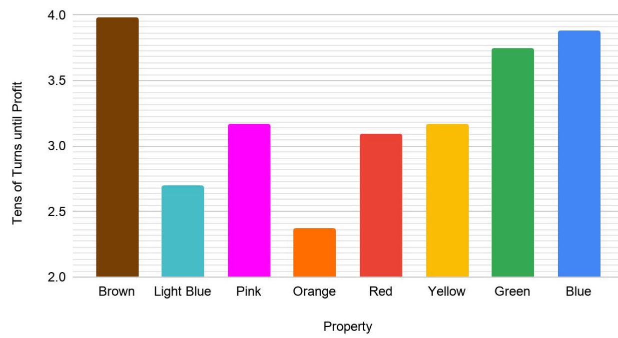


Figure 14: Tens of Turns until Profit (One Hotel) by Colour Set



3.2.2 Railroads

In Monopoly, railroads are an integral part of the game. They gain rent as players accumulate more of them throughout the game, eventually with a rent of $200 per landing on any railroad. In order to analyze the efficiency of using railroads, we must adapt our formula to determine the average probability of landing on the railroads. That is, because all railroads have a fixed rent, however there are many combinations in which players can own railroads, we use a slightly different formula to determine the efficiency of buying multiple railroads, where we average *X* values instead of *B* values:

$$P = A\,B\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right)$$

$$= A = \frac{P}{B\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right)}$$

As we know the probabilities each X value, we calculate that the average probability of landing on a single railroad is:

$$\frac{X_1 + X_2 + ... + X_n}{n}$$

$$= \frac{0.0244 + 0.0283 + 0.0295 + 0.0245}{4}$$

$$= \frac{0.1067}{4}$$

$$= 2.26675\%$$

Therefore, we can calculate the efficiency for any railroad, or group of railroads:

For 1 railroad: $A = \frac{200}{25\,(0.0226675)} \approx 352.93$ turns until the investment is made back

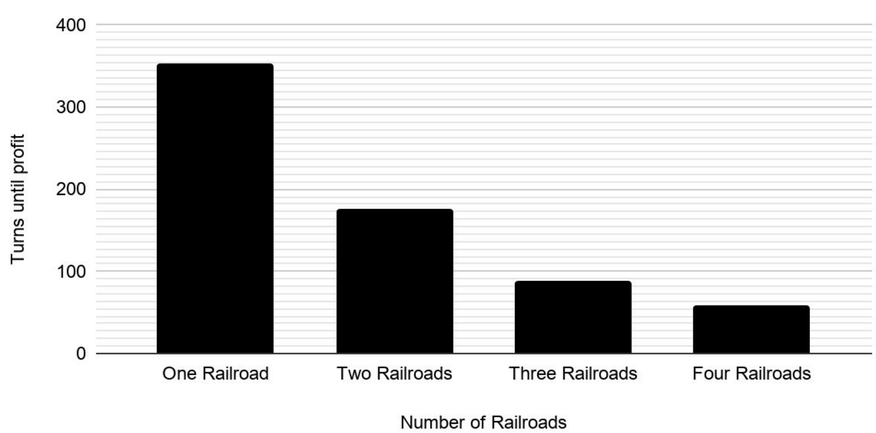
For 2 railroads: $A = \frac{400}{50\,(2 \times 0.0226675)} \approx 176.46$ turns until the investment is made back

For 3 railroads: $A = \frac{600}{100\,(3 \times 0.0226675)} \approx 88.23$ turns until the investment is made back

For 4 railroads: $A = \frac{800}{200\,(4 \times 0.0226675)} \approx 58.82$ turns until the investment is made back

With this in mind, we see that at its most efficient, railroads are about the same as using one or two houses on a property. Therefore, these become next to useless in the late game, however still have good early game applications. Figure 15 compares the railroads.

Figure 15: Number of turns until Profit based on how many railroads are controlled



Conclusion

In this paper, we have addressed and analyzed Craps, Blackjack, In-Between, and Monopoly using a variety of techniques ranging from simple probability calculations to massive computer programs designed to determine the probabilities. We have concluded several things regarding these games, to summarize: In craps, players are expected to lose money the slowest by betting against the point, therefore winning on a 7, and thus giving the player the statistically best option. In Blackjack, players can use the basic probability formula $q = \frac{n}{W}$ to find out how likely they are to go bust, and in In-Between, we determined when players should bet that the next card being drawn will be In-Between their previous two. In our third chapter, we examined how players can win at monopoly based on what properties are the most efficient at paying off the investment put into them. Here, we concluded that the best option is to put four houses on the Orange, Red, and Light Blue properties. While they are technically more efficient with one hotel, it is only just so, and by putting four houses on these properties, players effectively diminish the number of houses that their opponents can use on their properties. In the making of this report, we effectively demonstrated that mathematics can be applied to life-like scenarios to the advantage of those who apply it.

APPENDIX A

Monopoly simulation code

```
import random



#dice

die_one = [1, 2, 3, 4, 5,6]

die_two = [1, 2, 3, 4, 5,6]



position = 0

Doubles = 0



turn_log = []

Property_log = []



while True:

        #Rolls

        Roll_1 = random.choice(die_one)

        Roll_2 = random.choice(die_two)

        #NOT DOUBLES

        Move = Roll_1 + Roll_2
```

```
position = Move + position

if position > 39:

        position = position - 39

#Jail

if position == 30:

        position = 10

        #Chance

        elif position == 7 or position == 22 or position == 36:

                holder_1 = random.randint(1, 16)

                if holder_1 == 1:

                position = 0

elif holder_1 == 2:

        position = 24

elif holder_1 == 3:

        position = 11

elif holder_1 == 4:

        position = 10

elif holder_1 == 5:

        position = 39

elif holder_1 == 6:
```

```
        position = position - 3

elif holder_1 == 7:

        if position == 22:

        postition = 28

        else:

        position = 12

elif holder_1 == 8:

        if position == 7:

        position = 15

        elif position == 22:

        position = 25

        elif position == 36:

        position = 5

else:

position = position

#Community Chest

if position == 2 or position == 17 or position == 33:

        holder_2 = random.randint(1, 16)

        if holder_2 == 1:

        position = 0

        elif holder_2 == 2:
```

```
                print(position)

                position = 10

        else:

        position = position

        #Doubles

        if Roll_1 == Roll_2:

        Doubles = Doubles + 1

        if Doubles == 3:

                position = 10

                Doubles = 0

        elif Roll_1 != Roll_2:

        Doubles = 0

        turn_log.append(position)

        if len(turn_log) >= 1000000:

                break

        print(position)

        Property_log.append(position)

print("Property Stage")



Properties = ["Go", "Mediterranean", "Community Chest - 1", "Baltic", "Income Tax", "Reading Railroad",
"Oriental", "Chance - 1", "Vermont", "Connecticut", "Jail", "St. Charles", "Electric Company",
"States", "Virginia", "Pennsylvania Railroad", "St. James", "Community Chest - 2", "Tennessee", "New
```

```
York", "Free Parking", "Kentucky", "Chance - 2", "Indiana", "Illinois", "B&O Railroad", "Atlantic",
"Ventnor", "Water Works", "Marvin Gardens", "Go to Jail", "Pacific", "North Carolina", "Community Chest
- 3", "Pennsylvania", "Short Line Railroad", "Chance - 3", "Park Place", "Luxury Tax", "Boardwalk"]


n = 0

m = 0


while m < 39:

        n = n + 1

        m = m + 1

        print(Properties[n])

        print((Property_log.count(m))/1000000)
```

APPENDIX B

Derivation of Percentages for Monopoly

By individual property

Mediterranean Avenue:

$60 = A\ (0.0219)\,(2)$

$= \quad 60 = 0.0438\,A$

$= \quad 1369.86 = A$

Baltic Avenue:

$60 = A\,(0.0226)\,(4)$

$= \quad 60 = 0.0904\,A$

$= \quad 663.72 = A$

Oriental

$100 = A\,(0.0139)\,(6)$

$= \quad 100 = 0.0392\,A$

$= \quad 718.39 = A$

Vermont

$100 = A\,(0.0235)\,(6)$

$= \quad 100 = 0.141\,A$

$= \quad 709.22 = A$

St. James

Connecticut

$120 = A\,(0.0232)\,(8)$

$= \quad 120 = 0.1857\,A$

$= \quad 646.55 = A$

St. Charles

$140 = A\,(0.0275)\,(10)$

$= \quad 140 = 0.275\,A$

$= \quad 509.09 = A$

States

$140 = A\,(0.0239)\,(10)$

$= \quad 140 = 0.239\,A$

$= \quad 585.77 = A$

Virginia

$160 = A\,(0.0252)\,(12)$

$= \quad 160 = 0.3024\,A$

$= \quad 529.1 = A$

Indiana

$180 = A\,(0.0285)\,(14)$

$= \quad 180 = 0.399\,A$

$= \quad 451.13 = A$

$220 = A\,(0.0276)\,(18)$

$= \quad 220 = 0.4968\,A$

$= \quad 442.83 = A$

Tennessee

$180 = A\,(0.0299)\,(14)$

$= \quad 180 = 0.4186\,A$

$= \quad 430 = A$

Illinois

$240 = A\,(0.0324)\,(20)$

$= \quad 240 = 0.648\,A$

$= \quad 370.37 = A$

New York

$200 = A\,(0.0313)\,(16)$

$= \quad 200 = 0.5008\,A$

$= \quad 399.36 = A$

Atlantic

$260 = A\,(0.0279)\,(22)$

$= \quad 260 = 0.6138\,A$

$= \quad 423.59 = A$

Kentucky

$220 = A\,(0.0288)\,(18)$

$= \quad 220 = 0.5184\,A$

$= \quad 424.38 = A$

Ventnor

$260 = A\,(0.0275)\,(22)$

$= \quad 260 = 0.605\,A$

$= \quad 429.75 = A$

Marvin Gardens

$$280 = A\,(0.027)\,(24)$$

$$= \quad 280 = 0.648\,A$$

$$= \quad 432.1 = A$$

Pennsylvania

$$320 = A\,(0.0253)\,(28)$$

$$= \quad 320 = 0.7084\,A$$

$$= \quad 451.72 = A$$

Pacific

$$300 = A\,(0.0275)\,(26)$$

$$= \quad 300 = 0.715\,A$$

$$= \quad 419.58 = A$$

Park Place

$$350 = A\,(0.0222)\,(35)$$

$$= \quad 350 = 0.777\,A$$

$$= \quad 450.45 = A$$

North Carolina

$$300 = A\,(0.0268)\,(26)$$

$$= \quad 300 = 0.6968\,A$$

$$= \quad 430.54 = A$$

Boardwalk

$$400 = A\,(0.0271)\,(50)$$

$$= \quad 400 = 1.355\,A$$

$$= \quad 299.66 = A$$

By colour set

Brown properties

$$120 = A\,(0.0445)\left(\tfrac{4+8}{2}\right)$$

$= \quad 120 = 0.0267\,A$

$= \quad 449.44 = A$

Red properties

$$680 = A\,(0.0888)\left(\tfrac{36+36+40}{3}\right)$$

$= \quad 680 = 3.3152\,A$

$= \quad 205.12 = A$

Light Blue properties

$$320 = A\,(0.0699)\left(\tfrac{12+12+16}{3}\right)$$

$= \quad 320 = 0.932\,A$

$= \quad 343.35 = A$

Yellow properties

$$800 = A\,(0.0824)\left(\tfrac{44+44+48}{3}\right)$$

$= \quad 800 = 3.7355\,A$

$= \quad 214.16 = A$

Pink properties

$$440 = A\,(0.0766)\left(\tfrac{20+20+24}{3}\right)$$

$= \quad 440 = 1.6341\,A$

$= \quad 269.26 = A$

Green properties

$$920 = A\,(0.0796)\left(\tfrac{52+52+56}{3}\right)$$

$= \quad 920 = 4.245\overline{3}\,A$

$= \quad 216.71 = A$

Orange properties

$$560 = A\,(0.0897)\left(\tfrac{28+28+32}{3}\right)$$

$= \quad 560 = 2.6312\,A$

$= \quad 212.83 = A$

Blue properties

$$750 = A\,(0.0493)\left(\tfrac{70+100}{2}\right)$$

$= \quad 750 = 4.1905\,A$

$= \quad 178.9763$

By colour set with one house

Brown properties

$$220 = A\,(0.0445)\left(\frac{10+20}{2}\right)$$

$$= \quad 220 = 0.6675\,A$$

$$= \quad 329.59 = A$$

Red properties

$$1130 = A\,(0.0888)\left(\frac{90+90+100}{3}\right)$$

$$= \quad 1130 = 8.288\,A$$

$$= \quad 136.34 = A$$

Light Blue properties

$$470 = A\,(0.0699)\left(\frac{30+30+40}{3}\right)$$

$$= \quad 470 = 2.3499\,A$$

$$= \quad 201.72 = A$$

Yellow properties

$$1250 = A\,(0.0824)\left(\frac{110+110+120}{3}\right)$$

$$= \quad 1250 = 9.3387\,A$$

$$= \quad 133.85 = A$$

Pink properties

$$740 = A\,(0.0766)\left(\frac{50+50+60}{3}\right)$$

$$= \quad 740 = 4.0853\,A$$

$$= \quad 181.14 = A$$

Green properties

$$1520 = A\,(0.0796)\left(\frac{130+130+150}{3}\right)$$

$$= \quad 1520 = 10.8787\,A$$

$$= \quad 139.72 = A$$

Orange properties

$$860 = A\,(0.0897)\left(\frac{70+70+80}{3}\right)$$

$$= \quad 860 = 6.578\,A$$

$$= \quad 130.74 = A$$

Blue properties

$$1150 = A\,(0.0493)\left(\frac{175+200}{2}\right)$$

$$= \quad 1150 = 9.2438\,A$$

$$= \quad 124.41 = A$$

By colour set with two houses

Brown properties

$$320 = A\,(0.0445)\left(\frac{30 + 60}{2}\right)$$

$= \quad 320 = 2.003\,A$

$= \quad 159.8 = A$

Red properties

$$1580 = A\,(0.0888)\left(\frac{250 + 250 + 300}{3}\right)$$

$= \quad 1580 = 23.68\,A$

$= \quad 66.72 = A$

Light Blue properties

$$620 = A\,(0.0699)\left(\frac{90 + 90 + 100}{3}\right)$$

$= \quad 620 = 6.524\,A$

$= \quad 95.03 = A$

Yellow properties

$$1700 = A\,(0.0824)\left(\frac{330 + 330 + 360}{3}\right)$$

$= \quad 1700 = 28.016\,A$

$= \quad 60.68 = A$

Pink properties

$$1040 = A\,(0.0766)\left(\frac{150 + 150 + 180}{3}\right)$$

$= \quad 1040 = 12.256\,A$

$= \quad 84.86 = A$

Green properties

$$2120 = A\,(0.0796)\left(\frac{390 + 390 + 450}{3}\right)$$

$= \quad 2120 = 32.636\,A$

$= \quad 64.96 = A$

Orange properties

$$1160 = A\,(0.0897)\left(\frac{200 + 200 + 220}{3}\right)$$

$= \quad 1160 = 18.538 = A$

$= \quad 62.57 = A$

Blue properties

$$1550 = A\,(0.0493)\left(\frac{600 + 800}{2}\right)$$

$= \quad 1550 = 27.115\,A$

$= \quad 57.16 = A$

By colour set with three houses

Brown properties

$$420 = A\,(0.0445)\left(\frac{90 + 180}{2}\right)$$

$$= \quad 420 = 6.0075\,A$$

$$= \quad 69.13 = A$$

Red properties

$$2030 = A\,(0.0888)\left(\frac{700 + 700 + 750}{3}\right)$$

$$= \quad 2030 = 63.64\,A$$

$$= \quad 31.9 = A$$

Light Blue properties

$$770 = A\,(0.0699)\left(\frac{270 + 270 + 300}{3}\right)$$

$$= \quad 770 = 19.572\,A$$

$$= \quad 39.34 = A$$

Yellow properties

$$2150 = A\,(0.0824)\left(\frac{800 + 800 + 850}{3}\right)$$

$$= \quad 2150 = 67.2933\,A$$

$$= \quad 31.95 = A$$

Pink properties

$$1340 = A\,(0.0766)\left(\frac{450 + 450 + 500}{3}\right)$$

$$= \quad 1340 = 35.7466\,A$$

$$= \quad 37.49 = A$$

Green properties

$$2720 = A\,(0.0796)\left(\frac{900 + 900 + 1000}{3}\right)$$

$$= \quad 2720 = 74.2933\,A$$

$$= \quad 36.61 = A$$

Orange properties

$$1460 = A\,(0.0897)\left(\frac{550 + 550 + 600}{3}\right)$$

$$= \quad 1460 = 50.83\,A$$

$$= \quad 28.72 = A$$

Blue properties

$$2550 = A\,(0.0493)\left(\frac{1100 + 1400}{2}\right)$$

$$= \quad 2550 = 61.625\,A$$

$$= \quad 41.38 = A$$

By colour set with four houses

Brown properties

$$520 = A\,(0.0445)\left(\tfrac{160 + 320}{2}\right)$$

= $520 = 10.68\,A$

= $48.69 = A$

Red properties

$$2480 = A\,(0.0888)\left(\tfrac{875 + 875 + 925}{3}\right)$$

= $2480 = 78.18\,A$

= $31.3 = A$

Light Blue properties

$$920 = A\,(0.0699)\left(\tfrac{400 + 400 + 480}{3}\right)$$

= $920 = 29.125\,A$

= $31.59 = A$

Yellow properties

$$2600 = A\,(0.0824)\left(\tfrac{975 + 975 + 1025}{3}\right)$$

= $2600 = 681.7133\,A$

= $31.82 = A$

Pink properties

$$1640 = A\,(0.0766)\left(\tfrac{625 + 625 + 700}{3}\right)$$

= $1640 = 49.79\,A$

= $32.9383 = A$

Green properties

$$3330 = A\,(0.0796)\left(\tfrac{1100 + 1100 + 1200}{3}\right)$$

= $3330 = 90.2133\,A$

= $36.91 = A$

Orange properties

$$1760 = A\,(0.0897)\left(\tfrac{750 + 750 + 800}{3}\right)$$

= $1760 = 68.77\,A$

= $25.6 = A$

Blue properties

$$2950 = A\,(0.0493)\left(\tfrac{1300 + 1700}{2}\right)$$

= $2950 = 73.95\,A$

= $39.89 = A$

By colour set with one hotel

Brown properties

$$620 = A\,(0.0445)\left(\tfrac{250 + 450}{2}\right)$$

$$= \quad 620 = 15.575\,A$$

$$= \quad 39.81 = A$$

Red properties

$$2930 = A\,(0.0888)\left(\tfrac{1050 + 1050 + 1100}{3}\right)$$

$$= \quad 2930 = 94.72\,A$$

$$= \quad 30.93 = A$$

Light Blue properties

$$1070 = A\,(0.0699)\left(\tfrac{550 + 550 + 600}{3}\right)$$

$$= \quad 1070 = 39.61\,A$$

$$= \quad 27.01 = A$$

Yellow properties

$$3050 = A\,(0.0824)\left(\tfrac{1150 + 1150 + 1200}{3}\right)$$

$$= \quad 3050 = 96.1\overline{3}\,A$$

$$= \quad 31.73 = A$$

Pink properties

$$1940 = A\,(0.0766)\left(\tfrac{750 + 750 + 900}{3}\right)$$

$$= \quad 1940 = 61.28\,A$$

$$= \quad 31.66 = A$$

Green properties

$$3930 = A\,(0.0796)\left(\tfrac{1275 + 1275 + 1400}{3}\right)$$

$$= \quad 3930 = 104.81\,A$$

$$= \quad 37.5 = A$$

Orange properties

$$2060 = A\,(0.0897)\left(\tfrac{950 + 950 + 800}{3}\right)$$

$$= \quad 2060 = 86.71\,A$$

$$= \quad 23.76 = A$$

Blue properties

$$3350 = A\,(0.0493)\left(\tfrac{1500 + 2000}{2}\right)$$

$$= \quad 3350 = 86.275\,A$$

$$= \quad 38.83 = A$$

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